The Innkeeper's Problem: How to Keep from Yawning at Night at the Inn

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This problem is known by other names and guise.

Introduced as “Mismatched Cards” by Montmart, 1878

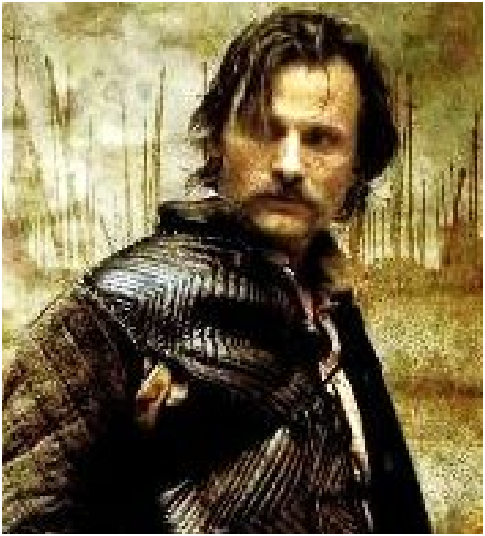
Numbers placed all in wrong slots

Letters stuffed all wrongly into envelopes

Instead, imagine an innkeeper with numbered Room Keys and Hooks to hang them on—

One key per hook and all placed wrongly!!

El Capitán Alatriste no conocía el posadero



Innkeeper wonders: What if I placed keys randomly on hooks?

- "A test of intellect, something to while the time away...."
- So he shakes the keys in a box and places them **randomly** on the hooks.
- Sometimes some keys wind up on the right hooks.
- But **sometimes no key** winds up on the right hook.
- Question: How probable is that—no key on the right hook?
- A very good question for a Math lecture!

The Innkeeper made a clever observation:

Rooms	All Ways	Ways for Total Mismatch	Probability
1	1	0	$\uparrow$ 0.0
2	2	1	$\downarrow$ 0.5
3	6	2	$\uparrow$ 0.3333
4	24	9	$\downarrow$ 0.3750
5	120	44	$\uparrow$ 0.3667
6*	720	265	$\downarrow$ 0.3681
7*	5040	1854	$\uparrow$ 0.3679

Table: With *miraculous* patience, he reckoned: the number of rooms in an inn and, for each number, the various ways room keys can be hung on hooks, and the probability of a total mismatch. Odd-room probabilities increase, even-room probabilities decrease, squeezing toward ≈ 0.368. Inference: Same probability must hold for 100 rooms!!!

Here we go, but first—some reminders:

- Probability = $\frac{m}{n}$ [for n equally likely events]
- Factorials: $m! = 1 \cdot 2 \cdot 3 \cdots m$
- Binomial coefficients: $\binom{m}{n} = \frac{m!}{n!(m-n)!}$ [m-choose-n]
- Polynomial Multiplication: [*Convolution* of coefficients]

$(a + bx + cx^2 + \cdots)(\alpha + \beta x + \gamma x^2 + \cdots) =$

$a\alpha +$

$(a\beta + b\alpha)x +$

$(a\gamma + b\beta + c\alpha)x^2 + \cdots$

Here are some other things we'll use:

Infinite series. Think of these as very large polynomials:

- $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$ for $x \in (-1, 1)$
- $\mathbf{e}^{-x} = 1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \dfrac{x^4}{4!} \pm \cdots$ for any $x \in \mathbb{R}$
- where $\mathbf{e} = 2.718281828\cdots$, base of natural logarithms

These series give accurate approximations to the functions they represent. And they can be multiplied like polynomials!

BTW, How many digits of e do you know?

e = 2.
7182818284 5904523536 0287471352 6624977572
4709369995 9574966967 6277240766 3035354759
4571382178 5251664274 2746639193 2003059921
8174135966 2904357290 0334295260 5956307381
3232862794 3490763233 8298807531 9525101901
1573834187 9307021540 8914993488 4167509244
7614606680 8226480016 8477411853 7423454424
3710753907 7744992069 5517027618 3860626133
1384583000 7520449338 2656029760 6737113200
7093287091 2744374704 7230696977 2093101416
9283681902 5515108657 4637721112 5238978442
. . . . . .

The decimal representation is non-repeating.

We can multiply the series like polynomials:

$$e^{-x} \cdot \frac{1}{1-x} \; =$$

$$\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \pm \cdots\right) \quad \times$$

$$\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \quad =$$

$$1 + \left(1 - \frac{1}{1!}\right)x + \left(1 - \frac{1}{1!} + \frac{1}{2!}\right)x^2 + \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)x^3 + \cdots$$

We will use this result.

The innkeeper enumerated permutations by hand—we do it mathematically.

Let k be the number of rooms at an inn,

p_k be the number of ways you can place all k keys incorrectly.

If there are n keys and hooks, then there are $\binom{n}{k} p_{n-k}$ ways to place **exactly** k keys correctly.

So, there are $\sum_{k=1}^{n} \binom{n}{k} p_{n-k}$ ways to place at least one key correctly—if you define $p_0 = 1$.

Now we can write p_n in a recursion formula.

As just shown, with n rooms at the inn, there are $\sum_{k=1}^{n} \binom{n}{k} p_{n-k}$ ways to get at least one key placed correctly.

Therefore, the total number of ways in which no key is placed correctly must be,

$$p_n = n! - \sum_{k=1}^{n} \binom{n}{k} p_{n-k}, \quad \text{or} \quad \sum_{k=0}^{n} p_k \binom{n}{n-k} = n!$$

Or, canceling the common $n!$: $$\sum_{k=0}^{n} \frac{p_k}{k!} \cdot \frac{1}{(n-k)!} = 1$$

The previous sum looks like a coefficient in a product of polynomials.

But what should the polynomials be? (Recall how polynomials are multiplied:)

Define the infinite series $P(x) = \sum_{k=0}^{\infty} \frac{p_k}{k!} x^k$ then multiply

$$P(x)e^x = \left(\frac{p_0}{0!} + \frac{p_1}{1!}x + \frac{p_2}{2!}x^2 + \cdots\right)\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots\right)$$

So the coefficient of x^n is the **convolution**: $\sum_{k=0}^{n} \frac{p_k}{k!} \cdot \frac{1}{(n-k)!}$

We want to solve the recursion formula:

$$\sum_{k=0}^{n} \frac{p_k}{k!} \cdot \frac{1}{(n-k)!} = 1$$

Solve for $p_0, p_1, p_2, \ldots$ recursively, i.e., one after another.

Just compare coefficients of x^n in the equation,

$$P(x)e^x = \left(\frac{p_0}{0!} + \frac{p_1}{1!}x + \frac{p_2}{2!}x^2 + \cdots\right)\left(1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots\right)$$

$= 1 + x + x^2 + x^3 + \cdots$ [This yields the recursion formula.]

Get ready: $P(x) = e^{-x}\left(1 + x + x^2 + x^3 + \cdots\right) = \dfrac{e^{-x}}{1-x}$

So here comes the magical solution!

Put it all together: $P(x) = \sum_{k=0}^{\infty} \frac{p_k}{k!} x^k = \frac{e^{-x}}{1-x} =$

$$1 + (1 - \frac{1}{1!})x + (1 - \frac{1}{1!} + \frac{1}{2!})x^2 + (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!})x^3 + \cdots$$

Equate coefficients of x^n, get probability of misplacing all n keys:

$$\frac{p_n}{n!} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{n!} \approx e^{-1}$$

So, $\frac{p_n}{n!} \to e^{-1} \approx \mathbf{0.368}$ [The innkeeper was right! Amazing!!]

Where did the idea for this solution come from?

The essential equation is: $$\sum_{k=0}^{\infty} \frac{p_k}{k!} x^k = \frac{e^{-x}}{1-x}$$

The function on the right-hand side is called a generating function for the coefficients on the left-hand side.

The generating function made it easy to find $\frac{p_k}{k!}$.

The Swiss mathematician Leonhard Euler introduced the idea of generating functions in “Several analytic observations on combinations” of 1741.

References

The graphic of Capitán Alatriste is from Wikipedia's "Alatriste the film," used here for an educational purpose.

Nemiroff and Bonnell's decimal expansion of e accessed at, `www.mu.org/~doug/exp/100000.html`

For different approaches to solving the innkeeper's problem (named differently), see:

Courant, Robbins and Stewart, What Is Mathematics? An Elementary Approach to Ideas and Methods, Oxford University Press, pp. 144–116

William Feller, An Introduction to Probability Theory and Its Applications, Vol 1 Ed 3, John Wiley & Sons, Inc., p. 100

For Euler, C. Edward Sandifer, *The Early Mathematics of Leonhard Euler*, p. 377.

The full text of the innkeeper's problem is on my web under "Mathematics Miscellany"

These slides are there, too:

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